

Prediction of Brake Disc Mechanical Behavior Using Artificial Neural Networks

1st Haşmet Çağrı Sezgen

Department of Mechanical Engineering
Necmettin Erbakan University
Konya, Türkiye
hasmetcagri.sezgen@erbakan.edu.tr

2nd Erdi Gülbahçe

Department of Mechatronics Engineering
KTO Karatay University
Konya, Türkiye
erdi.gulbahce@karatay.edu.tr

3rd Abdullah Çakan

Department of Mechanical Engineering
Konya Technical University
Konya, Türkiye
acakan@ktun.edu.tr

Abstract— This study presents an artificial neural network (ANN)-based surrogate modeling framework for predicting key performance metrics of automotive brake discs using data derived from finite element analyses. A dataset comprising 200 geometrically parameterized design points was generated via Finite Element Method (FEM) simulations, including static structural, transient thermal and modal analyses in ANSYS/Workbench. The target outputs, maximum von Mises stress, peak temperature, first natural frequency and total volume were used to train a multi-output ANN implemented in MATLAB. The trained model demonstrates strong predictive capability with high correlation coefficients ($R > 0.98$) and low Mean Absolute Percentage Error (MAPE $< 2.5\%$) across all metrics. Validation on six independent test cases confirmed the ANN's generalization ability and accuracy in replicating FEM results within acceptable error margins. The proposed approach significantly reduces the computational burden associated with conventional FEM simulations while maintaining reliable accuracy, making it an efficient tool for early-stage design evaluation and multi-objective optimization of brake disc components.

Keywords—Artificial Neural Network (ANN), Brake Disc Design, Finite Element Method (FEM), Multi-Physics Simulation, Surrogate Modeling

I. INTRODUCTION

Brake systems are among the most critical subsystems in ground vehicles, directly affecting driving safety. At the heart of this system is the brake disc (rotor), which plays a pivotal role by converting the vehicle's kinetic energy into heat through friction, thereby slowing down or stopping the vehicle [1, 2]. During braking, the disc is simultaneously subjected to intense mechanical loading (due to clamping pressure from the pads) and thermal loading (due to frictional heating). This combination of loads leads to sharp temperature gradients in the disc material and induces high thermal stress [1]. Such thermomechanical stresses can cause significant thermal expansion and cyclic fatigue damage; thermal fatigue has been identified as a primary cause of brake disc failure in high-speed and heavy-duty braking scenarios [3]. In addition to structural strength and thermal capacity, a brake disc's design must be optimized regarding multiple performance criteria including vibrational behavior, weight and manufacturability [4, 5]. Therefore, designing an effective brake disc is inherently a multi-objective, multi-physics problem.

The finite element method (FEM) remains the dominant technique for analyzing such complex behaviors, especially in automotive components exposed to multiple physical domains [6–8]. Commercial FEM platforms like ANSYS allow detailed simulations such as static structural analyses, modal analyses

for vibrational modes and transient thermal simulations. These high-fidelity simulations capture the complex behavior of brake discs accurately. However, this accuracy comes with high computational cost: solving coupled thermomechanical problems via FEM is time- and resource-intensive [9]. Particularly for parametric design studies or optimization, repeating FEM analyses for a large number of design points becomes impractical and delays the design cycle significantly [10, 11].

To overcome these limitations, surrogate modeling techniques have been widely explored [12]. In particular, artificial neural networks (ANNs) have emerged as powerful tools due to their capacity to approximate nonlinear, multivariate input–output mappings with high fidelity [13]. Once trained on high-quality simulation data, ANNs offer rapid predictions that can significantly accelerate the early design stages. Previous studies have shown the effectiveness of ANN-based surrogates in predicting single outputs such as thermal response [14], stress [15], or vibrational frequency [16]. For example, Roberts et al. modeled thermomechanical behavior of reinforced brake discs using ANN trained on FEM results [17] and Kuncy et al. applied ANNs to predict brake pad wear and coefficient of friction based on material and loading parameters [18]. Similar applications have been found in vibration prediction [19–21] and thermal fatigue analysis [22].

However, literature is dominated by studies predicting only a single physical characteristic—either mechanical strength or thermal behavior—of brake discs and lacks an approach capable of rapidly forecasting multiple engineering outputs simultaneously. To address this gap, the present work develops a multi-output artificial neural network (ANN) model trained on data obtained from finite element analyses, including maximum stress, first natural frequency, peak temperature and total volume. A design space defined parametrically by inner radius, outer radius and thickness variables was sampled to generate 200 custom DOE instances; each instance underwent static structural, modal and thermal analyses to form a comprehensive training dataset. The proposed ANN maintains high accuracy while reducing computation time to a matter of seconds compared to full FEM analyses, positioning itself as an effective accelerator for multi-criteria disc design. This methodology offers a novel contribution by simultaneously learning interdisciplinary outputs and accounting for manufacturable geometry ranges within practical engineering limits. The study's scope covers 200 parametrically defined design points for gray cast iron discs and semi-metallic composite pads under assumptions of material homogeneity and fixed boundary conditions.

Ultimately, the work compiles these FEM-derived mechanical, vibrational and thermal metrics through the DOE process, trains and evaluates the neural model's performance in comparison and provides recommendations to enhance design efficiency and multi-objective optimization.

II. FINITE ELEMENT ANALYSES

This study integrates Finite Element Analysis (FEA) with an Artificial Neural Network (ANN) model to efficiently predict the behavior of automotive brake discs. Comprehensive structural, modal and transient thermal analyses were performed in ANSYS Workbench on 200 parametrically defined disc geometries. These geometries were generated using a Design of Experiments (DOE) framework. The DOE points were systematically created based on three fundamental geometric parameters: disc thickness (P_1), outer radius (P_2) and inner radius (P_3), as illustrated in Fig. 1. The parameter ranges were defined based on typical brake disc dimensions reported in the literature and practical manufacturing constraints [23]. Specifically, P_1 ranged from 66 to 90 mm, P_2 from 124 to 150 mm and P_3 from 5 to 27 mm, reflecting standard design practices and allowable manufacturing measurements. The output vector comprises four distinct engineering metrics derived from FEA simulations: maximum von Mises stress (σ_{\max}) from static structural analysis, first natural frequency (f_1) from modal analysis, maximum temperature (T_{\max}) from transient thermal analysis and brake disc volume (V). These outputs serve as target variables for training the ANN model, enabling it to accurately predict multi-physics responses based on input geometry.

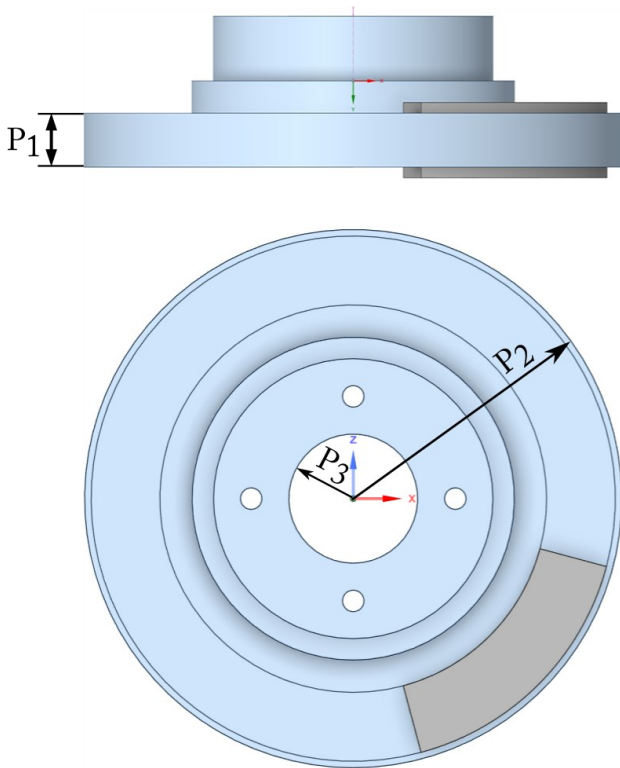


Fig. 1. DOE Parameters of Brake Disc

For each geometry, a consistent finite element model was developed in ANSYS/Workbench. The 3D disc geometry was parametrically defined and meshed using tetrahedral elements, with local refinement applied in stress-prone

regions. The mesh contained approximately 1366819 nodes and 976809 elements. The mesh model is shown in Fig. 2(b). For the accurate execution of these analyses, material properties were defined: Gray Cast Iron (EN-GJL-250) was assigned to the disc and a homogenized semi-metallic composite to the pads, with detailed properties listed in Tables 1 and 2. All simulations employed identical mesh settings, boundary conditions and material definitions to ensure consistency across structural, thermal and modal analyses.

TABLE I. BRAKE DISC MATERIAL PROPERTIES

Property	Value	Unit
Density (ρ)	7200	kg/m ³
Elastic Modulus (E)	110×10 ⁹	Pa (N/m ²)
Poisson's Ratio (ν)	0.28	-
Thermal Conductivity (k)	52	W/m·K
Specific Heat Capacity (Cp)	447	J/kg·K
Maximum Operating Temperature	~600	°C

TABLE II. BRAKE PAD MATERIAL PROPERTIES

Property	Value	Unit
Density (ρ)	3000	kg/m ³
Elastic Modulus (E)	7×10 ⁹	Pa (N/m ²)
Poisson's Ratio (ν)	0.32	-
Thermal Conductivity (k)	10	W/m·K
Specific Heat Capacity (Cp)	800	J/kg·K
Maximum Operating Temperature	500–600	°C

Each brake disc design was analyzed using a series of finite element simulations, including static structural, transient thermal and modal analyses, in order to generate a comprehensive and representative training dataset for the artificial neural network (ANN) model. These simulations were conducted for different geometric configurations to capture the influence of design parameters on the mechanical behavior of the brake disc under realistic operating conditions. The boundary conditions, loading scenarios and thermal assumptions applied in the FEM models were carefully selected to reflect emergency braking conditions typically encountered in automotive applications. These conditions were based on validated methodologies and standardized practices widely cited in the literature [23], ensuring that the simulation outputs closely resemble real-world physical behavior. By using a combination of mechanical and thermal analyses, the dataset provides a diverse set of response variables that enable the ANN model to learn complex input-output relationships with high fidelity.

In the static structural analysis, the primary objective was to determine the maximum von Mises stress (σ_{\max}) induced by braking loads. The disc was constrained using a rigid revolute joint at the hub region and a static pressure load of 10.495 MPa was applied to simulate the contact force from the brake pads. All boundary conditions are illustrated in Fig. 2(a). An angular velocity of 250 rad/s was applied to replicate the disc's rotational motion during braking and a friction coefficient of 0.22 was defined at the disc-pad interface.

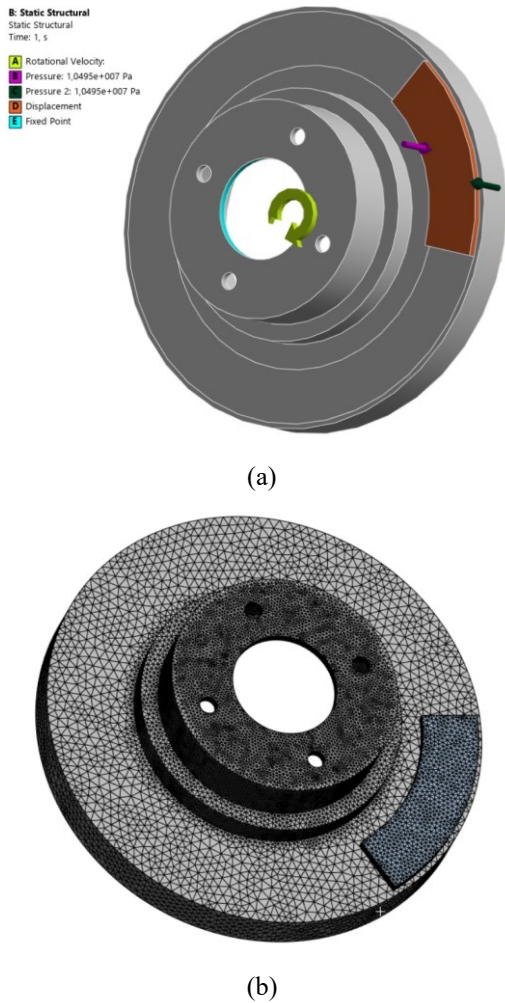


Fig. 2. Finite element modeling, (a) boundary conditions for static structural analysis, (b) mesh model for all analysis

A transient thermal analysis was performed to calculate the temperature distribution resulting from frictional heat generation. A heat flux of $1.54 \times 10^6 \text{ W/m}^2$ was applied to both sides of the disc surface. The initial temperature was set to $35 \text{ }^\circ\text{C}$ and convection with a film coefficient of $5 \text{ W/m}^2 \cdot \text{C}$ was applied to the outer surfaces to simulate natural convection. The simulation time was defined as 5 seconds and the maximum temperature (T_{max}) reached during this period was recorded.

To analyze the vibrational behavior of the brake disc, a free-free modal analysis was performed. In this analysis, the brake pad geometry was removed from the model to focus only on the disc's own dynamics. This made it easier to understand how the disc responds to vibration. The first bending natural frequency (f_1), which is important for evaluating possible noise and vibration issues in braking systems, was calculated as the main vibrational output.

The results of the finite element simulations for a representative design point are summarized in Fig. 3. Fig.3(a) presents the von Mises stress distribution obtained from the static structural analysis, illustrating that maximum stress concentrations occur primarily in the inner radius region adjacent to the braking surface. Fig. 3(b) depicts the temperature field from the transient thermal analysis, highlighting the localized heat accumulation during braking. Fig 3(c) shows the first mode shape extracted from the modal analysis, which was recorded as the fundamental natural

frequency (f_1) for each configuration. These graphical representations provide visual insight into the physical behavior of the system and support the numerical outputs used for training the ANN model. Additionally, the contour maps were archived to assist with interpretability and future validation processes.

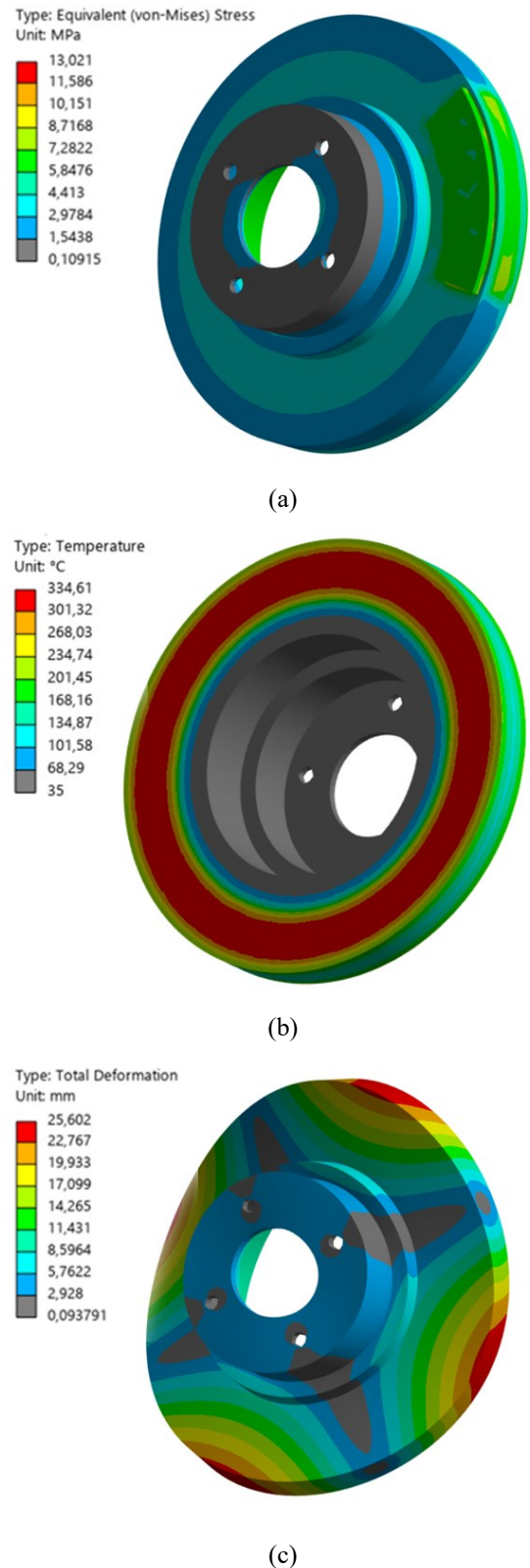


Fig. 3. Analysis results for a representative design point, (a) static structural analysis, (b) transient thermal analysis, (c) modal analysis

III. ARTIFICIAL NEURAL NETWORK

Artificial Neural Networks (ANNs) are widely used machine learning models that simulate the structure and learning mechanisms of the human brain. They consist of layers of interconnected processing units called artificial neurons, which collectively process and learn from data [24, 25]. ANNs operate under a supervised learning framework, where the model is trained using labeled data by minimizing the difference between predicted outputs and actual targets through iterative weight adjustments.

One of the most common ANN architectures is the Multi-Layer Perceptron (MLP) [26, 27], which includes at least one hidden layer between the input and output layers. The number of hidden layers and neurons is typically determined through trial and error based on the model's performance. A Multi-Layer Perceptron is a type of ANN that consists of at least three layers: an input layer, one or more hidden layers and an output layer, shown in Fig. 4.

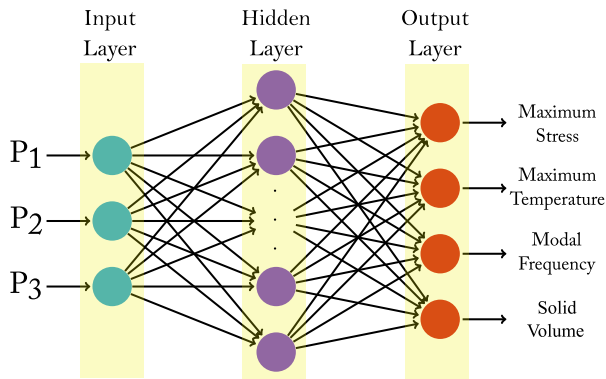


Fig. 4. ANN Structure of this study.

The ANN modeling process generally involves several key steps: data preparation, network architecture design, model training, performance evaluation and deployment. In this study, these steps were implemented using MATLAB, which provides a robust environment for developing and simulating ANN models. The dataset was randomly divided into training, validation and testing subsets, commonly using a 70–15–15% ratio [28, 29]. The Levenberg-Marquardt algorithm was used for efficient training [30]. The model specifications of the ANN are given Table 3.

TABLE III. THE MODEL SPECIFICATIONS OF THE ANN

Training Data (%)	70
Validation Data (%)	15
Testing Data (%)	15
Number of neurons	20
ANN Training Algorithm	Levenberg-Marquardt
Data Division Method	Random

Model performance was evaluated using Mean Squared Error (MSE) and the correlation coefficient (R), which are standard metrics for regression tasks. The mathematical formulations of MSE and R are presented in Eq. (1) and Eq. (2), respectively. These evaluations helped determine how accurately the trained ANN models could predict target values

and generalize new data. Specifically, $y_{\text{target}(i)}$ refers to the actual value of the i -th sample, $y_{\text{ANN}(i)}$ is the ANN-predicted value for that sample and N denotes the total number of samples in the dataset. A lower MSE indicates better prediction accuracy.

$$\text{MSE} = \frac{1}{N} \sum_{i=1}^N (y_{\text{target}(i)} - y_{\text{ANN}(i)})^2 \quad (1)$$

$$R = \sqrt{1 - \frac{\sum_{i=1}^N (y_{\text{target}(i)} - y_{\text{ANN}(i)})^2}{\sum_{i=1}^N (y_{\text{target}(i)})^2}} \quad (2)$$

IV. RESULTS AND DISCUSSIONS

This section presents the results of ANN-based prediction model. The ANN was trained using data generated from FEM analyses of various brake disc geometries to estimate key mechanical responses, including equivalent stress, temperature, deformation and solid volume. The model's performance was evaluated using correlation analysis, scatter plots and error histograms and further validated on test cases.

Correlation coefficient results for the ANN model are presented in Fig. 5. The scatter plots illustrate the relationship between the predicted outputs and the actual target values. As seen in the figure, the training data shows a strong match, which indicates that the ANN model has learned the patterns in the data accurately and makes reliable predictions.

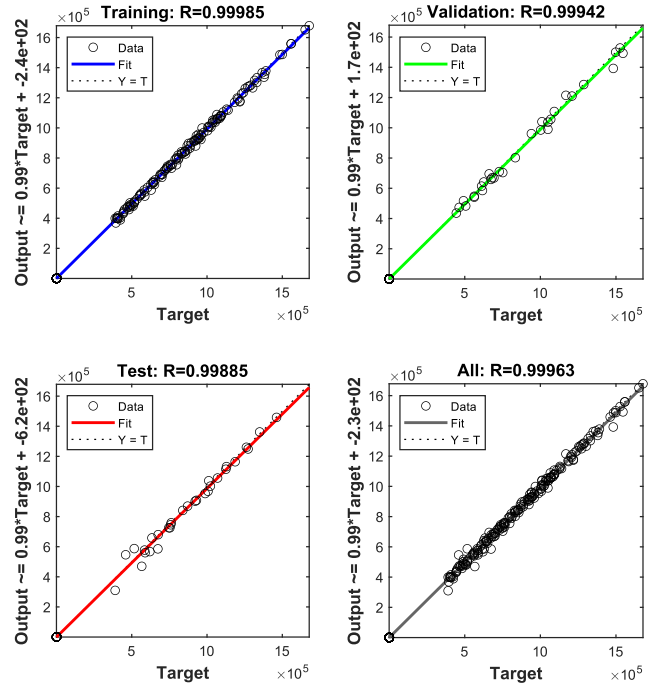


Fig. 5. Correlation coefficient results of this study.

The error histogram plot illustrates how the prediction errors are distributed across the dataset. As shown in Fig. 6, the errors for the training, validation and testing sets are approximately normally distributed and centered around zero. This suggests that the ANN model does not exhibit bias and performs consistently across different subsets of the data.

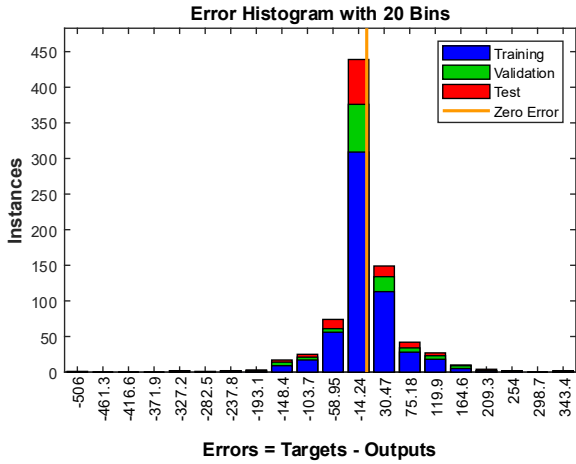


Fig. 6. The error histogram plot of this study.

Table 4. presents the geometric design parameters used in six independent test cases that were intentionally excluded from the training and validation stages of the ANN model. These cases serve as a control group to assess the generalization performance of the network beyond the data it has been exposed to during training.

TABLE IV. GEOMETRIC DESIGN PARAMETERS USED IN TEST CASES

Case No	(P ₁) Disc Thickness (mm)	(P ₂) Outer Radius (mm)	(P ₃) Inner Radius (mm)
1	21	140	41
2	19	128	35
3	21	124	41
4	21	131	43
5	19	124	38
6	10	148	33

Table 5 shows the comparison between the ANN-predicted results and the FEM simulations for the same test cases. The evaluated outputs include maximum equivalent stress, maximum temperature, total deformation and solid volume. The high consistency between the ANN and FEM values across all metrics supports the robustness of the trained model. This close agreement highlights the predictive accuracy of the trained ANN model and demonstrates its potential to serve as a reliable surrogate model for estimating brake disc behavior under various geometric configurations.

TABLE V. COMPARISON BETWEEN THE ANN-PREDICTED RESULTS AND FEM SIMULATIONS FOR THE TEST CASES

Case No	Maximum Stress (MPa)		Maximum Temperature (°C)	
	ANSYS	ANN	ANSYS	ANN
1	14.1516	14.3584	351.2440	348.4718
2	13.6112	13.4517	359.2663	361.4475
3	14.1000	13.4867	350.9480	354.6319
4	13.6671	14.0110	347.6618	350.1808
5	13.2482	12.8954	363.2302	366.1254
6	14.9125	14.7367	526.4587	525.0571

Case No	Modal Frequency (Hz)		Solid Volume (mm ³)	
	ANSYS	ANN	ANSYS	ANN
1	1216.5625	1213.31	1085524.76	1085706.15
2	1487.2733	1490.89	879606.23	879869.00
3	1380.1767	1426.24	806001.38	807034.31
4	1271.6237	1291.24	903173.83	902912.58
5	1530.5698	1542.42	787660.65	787770.07
6	727.1660	725.33	732422.46	732729.36

To quantitatively measure the prediction accuracy, Table 6 reports the percentage error for each output parameter and test case. These error margins confirm that the ANN model provides reliable predictions for previously unseen input configurations.

TABLE VI. PERCENTAGE ERRORS BETWEEN ANN PREDICTIONS AND FEM RESULTS FOR EACH OF THE TEST CASES

Case No	% Error			
	Maximum Stress (MPa)	Maximum Temperature (C)	Frequency (Hz)	Solid Volume (mm ³)
1	1.46	0.79	0.27	0.02
2	1.17	0.61	0.24	0.03
3	4.35	1.05	3.34	0.13
4	2.52	0.72	1.54	0.03
5	2.66	0.80	0.77	0.01
6	1.18	0.27	0.25	0.04
MAPE	2.22	0.71	1.07	0.04

After the ANN model was developed, its predictive performance was evaluated by comparing the ANN predictions and ANSYS simulation results for six experimental cases that were not included in the training dataset. The comparison was conducted using the Mean Absolute Percentage Error (MAPE) to quantify the accuracy of the model.

$$MAPE = \frac{100\%}{n} \sum_{i=1}^n \left| \frac{y_i - \hat{y}_i}{y_i} \right| \quad (3)$$

V. CONCLUSION

This work introduces a multi-output artificial neural network (ANN) model trained on high-fidelity FEM data to accelerate the evaluation and optimization of brake disc designs. By integrating structural, thermal and vibrational analyses into a unified surrogate model, the approach enables simultaneous prediction of multiple engineering metrics with high accuracy. The ANN model achieved strong agreement with FEM results across both training and independent test sets, with error margins generally below 2.5%. Compared to traditional simulation workflows, the ANN-based approach reduces computational time from hours to seconds, providing significant advantages in iterative design tasks, sensitivity analyses and early-stage prototyping. The methodology holds

promise for future applications in lightweighting, NVH optimization and smart braking systems, especially when coupled with evolutionary optimization algorithms or real-time design platforms. Overall, the study demonstrates the feasibility and effectiveness of machine learning models in replacing or complementing computationally expensive simulations in automotive component design.

ACKNOWLEDGMENT

This work is supported by The Scientific and Technological Research Council of Türkiye, grant program for participation in scientific meetings abroad.

REFERENCES

- [1] C. B. Saiz, T. Ingrassia, V. Nigrelli and V. Ricotta, "Thermal stress analysis of different full and ventilated disc brakes," *Fracture and Structural Integrity*, vol. 9, no. 34, 2015.
- [2] M. Eriksson, F. Bergman and S. Jacobson, "On the nature of tribological contact in automotive brakes," *Wear*, vol. 252, no. 1-2, pp. 26-36, 2002.
- [3] B.-c. Goo and C.-h. Lim, "Thermal fatigue of cast iron brake disk materials," *Journal of mechanical science and technology*, vol. 26, no. 6, pp. 1719-1724, 2012.
- [4] Y. Patil, S. Khan, S. Iqbal, A. Bankar and M. Patil, "Brake Squeal Analysis using Finite Element Analysis Method," *International Journal of Engineering Sciences*, vol. 13, pp. 72-81.
- [5] F. Talati and S. Jalalifar, "Analysis of heat conduction in a disk brake system," *Heat and mass transfer*, vol. 45, no. 8, pp. 1047-1059, 2009.
- [6] S. Abdullah, N. Al-Asady, A. Ariffin and M. Rahman, "A review on finite element analysis approaches in durability assessment of automotive components," *Journal of Applied Sciences*, vol. 8, no. 12, pp. 2192-2201, 2008.
- [7] S. K. Behera, S. P. Singh and B. K. Behera, "A review of multiscale FE modelling of lightweight composite for automotive part," *Proceedings of the Institution of Mechanical Engineers, Part D: Journal of Automobile Engineering*, p. 09544070251349977, 2024.
- [8] A. P. Prasetyono, A. Yudianto and I. W. Adiyasa, "Lightweight Design and Finite Element Analysis of Brake Lever for Motorcycle Application," *ComTech: Computer, Mathematics and Engineering Applications*, vol. 14, no. 1, pp. 21-32, 2023.
- [9] A. Belhocine and M. Bouchetara, "Thermomechanical behaviour of dry contacts in disc brake rotor with a grey cast iron composition," *Transactions of the Indian Institute of Metals*, vol. 65, no. 3, pp. 231-238, 2012.
- [10] A. Forrester, A. Sobester and A. Keane, *Engineering design via surrogate modelling: a practical guide*. John Wiley & Sons, 2008.
- [11] R. Jin, W. Chen and T. W. Simpson, "Comparative studies of metamodelling techniques under multiple modelling criteria," *Structural and multidisciplinary optimization*, vol. 23, no. 1, pp. 1-13, 2001.
- [12] S. Jayasinghe et al., "A review on the applications of artificial neural network techniques for accelerating finite element analysis in the civil engineering domain," 2025.
- [13] S. Haykin, *Neural networks and learning machines*, 3/E. Pearson Education India.
- [14] K.-T. Yang, "Artificial neural networks (ANNs): a new paradigm for thermal science and engineering," 2008.
- [15] R. Haj-Ali, H.-K. Kim, S. W. Koh, A. Saxena and R. Tummala, "Nonlinear constitutive models from nanoindentation tests using artificial neural networks," *International Journal of Plasticity*, vol. 24, no. 3, pp. 371-396, 2008.
- [16] M. R. S. Reddy, B. S. Reddy, V. N. Reddy and S. Sreenivasulu, "Prediction of natural frequency of laminated composite plates using artificial neural networks," *Engineering*, vol. 4, no. 6, pp. 329-337, 2012.
- [17] S. Roberts, J. Kusiak, Y. Liu, A. Forcellese and P. Withers, "Prediction of damage evolution in forged aluminium metal matrix composites using a neural network approach," *Journal of Materials Processing Technology*, vol. 80, pp. 507-512, 1998.
- [18] I. K. Kuncy, A. Abugh and T. L. Tyovenda, "Prediction of wear and friction coefficient of brake pads developed from palm kernel fibres using artificial neural network," *Journal of Engineering Studies and Research*, vol. 20, no. 3, p. 45, 2014.
- [19] N. Grochevaia, "Prediction of Brake Squeal: A Deep Learning Approach Analysis by Means of Recurrent Neural Networks," 2020.
- [20] M. Stender and N. Hoffmann, "Deep learning for predicting brake squeal," in *International Conference on Noise and Vibration Engineering (ISMA 2020) and International Conference on Uncertainty in Structural Dynamics (USD 2020)*, 2020, pp. 3327-3337.
- [21] M. Stender, M. Tiedemann, D. Spieler, D. Schoepflin, N. Hoffmann and S. Oberst, "Deep learning for brake squeal: vibration detection, characterization and prediction," *arXiv preprint arXiv:2001.01596*, 2020.
- [22] T. H. Kori, A. J. Santhosh, D. M. Yona, N. Ashok, G. Thilak and A. J. A. Afresa, "Mathematical modeling and prediction of pit to crack transition under cyclic thermal load using artificial neural network," *Materials Today: Proceedings*, 2023.
- [23] A. Durgude, A. Vipradas, S. Kishore and S. Nimse, "Design optimisation of brake disc geometry," *MAE 598 - 2016 - 11*, Final Report, 2016.
- [24] H. Jin, Y.-G. Kim, Z. Jin, A. A. Rushchite and A. S. Al-Shati, "Optimization and analysis of bioenergy production using machine learning modeling: Multi-layer perceptron, Gaussian processes regression, K-nearest neighbors and Artificial neural network models," *Energy Reports*, vol. 8, pp. 13979-13996, 2022.
- [25] Y. Mu and L. Sun, "Catalyst optimization design based on artificial neural network," *Asian J. Res. Comput. Sci.*, vol. 13, no. 2, pp. 1-12, 2022.
- [26] R. Roy and A. K. Gupta, "Data-driven prediction of flame temperature and pollutant emission in distributed combustion," *Applied Energy*, vol. 310, p. 118502, 2022.
- [27] A. Sharma, K. Goswami, V. Jindal and R. Gupta, "A road map to artificial neural network," in *Recurrent Neural Networks: CRC Press*, 2022, pp. 3-21.
- [28] S. A. Abdel-Razek, H. S. Marie, A. Alshehri and O. M. Elzeki, "Energy efficiency through the implementation of an AI model to predict room occupancy based on thermal comfort parameters," *Sustainability*, vol. 14, no. 13, p. 7734, 2022.
- [29] V. R. Joseph, "Optimal ratio for data splitting," *Statistical Analysis and Data Mining: The ASA Data Science Journal*, vol. 15, no. 4, pp. 531-538, 2022.
- [30] H. P. Gavin, "The Levenberg-Marquardt algorithm for nonlinear least squares curve-fitting problems," *Department of Civil and Environmental Engineering Duke University August*, vol. 3, pp. 1-23, 2019.